1- Linear Systems

(1)Solve the following linear systems by Gauss, Crammer, inverse methods, if possible:

(a) $x + y = 3$
$3x - y = 1$

(b) $2x + y = 6$
$3x - y = 4$

(c) $x + y + z = 5$
$2x - y + z = 2$
$2x + 2y - z = 4$

(d) $x + y - z = 3$
$x - y + 2z = 5$
$2x + 2y - 2z = 6$

(e) $2x + y + 2z = 8$
$x - y + z = 1$
$x + y + 2z = 7$

(g) $x_1 + x_2 - x_3 + x_4 = 4$
$2x_1 + x_2 + x_3 - x_4 = 3$
$x_1 - x_2 + 2x_3 + x_4 = 6$
$-x_1 + x_2 + x_3 - x_4 = 0$

(2)Solve the following linear system:

$x_1 + x_2 + x_3 + x_4 = 10, \quad x_1 - x_2 + 2x_3 - x_4 = 1$
$2x_1 + x_2 - x_3 + x_4 = 5, \quad x_1 + x_2 + 2x_3 - 2x_4 = 1$

(3)A liquid product exists in three concentrations:

The first of concentration: 1 mg /bottle, the second of concentration: 2 mg / bottle and the third type of concentration: 4 mg / bottle.

If we want to prepare 10 bottles of concentration 3 mg / bottle by mixing whole bottles of each type. Find all possible solutions.

(4)A liquid product exists in three concentrations:

The first of concentration: 1 mg /bottle, the second of concentration: 3 mg / bottle and the third type of concentration: 4 mg / bottle.

If we want to prepare 10 bottles of concentration 2 mg / bottle by mixing whole bottles of each type. Find all possible solutions.

(5)A liquid product exists in three concentrations:

The first of concentration: 3 mg /bottle, the second of concentration: 2 mg / bottle and the third type of concentration: 5 mg / bottle.

If we want to prepare 15 bottles of concentration 2.5 mg / bottle by mixing whole bottles of each type. Find all possible solutions.

(6)By the iterative method, solve the following linear systems:

(a) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -3 \\ 1 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -2 & 1 \\ -1 & 2 & 1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & -1 & 2 & -1 \\ 1 & 2 & -1 & -1 \\ 1 & -2 & 2 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 7 \\ 8 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & -3 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & -2 & 3 & 1 \\ 2 & -2 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 8 \\ 2 \end{bmatrix}$

2- Linear Programming

(1)Determine the convex sets among the following sets:

$A = \{(x, y) : 4x^2 + 9y^2 = 36\}$

$B = \{(2,3),(4,7)\}$

$C = \{(x, y) : x^2 + y^2 \le 4\}$

$D = \{(x, y) : x + y = 5\}$

$E = \{(x, y) : x^2 + y^2 \ge 1 \text{ and } x^2 + y^2 \le 9\}$

$F = \{(x, y) : x + y \le 5, -x + y \le 2, x - y \le 3\}$

(2)Solve the following LP problems graphically:

(i) maximize $f = x + y$

s.t $5x + 2y \le 180$

$3x + 3y \le 135$

$x, y \ge 0$

(ii) maximize $f = 3x + 2y$

s.t $x + 2y \le 76$

$2x + y \le 104$

$x, y \ge 0$

(iii) maximize $f = x + 2y$

s.t $x + 2y \ge 2$

$-x + 3y \le 6$

$x - y \le 4$

$x, y \ge 0$

(iv) minimize $f = 2x + 3y$

s.t $x + y \ge 3$

$-x + y \le 1$

$x - y \le 4$

$x, y \ge 0$

(v) maximize $f = 2x + 5y$

s.t $3x + 5y \le 30$

$x \le 4$

$y \le 5$

$x, y \ge 0$

(vi) minimize $f = 4x + 5y + 8$

s.t $x - y = 2$

$x + y = 4$

$2x + y \le 10$

$x, y \ge 0$

(3)Solve the following LP problems :

(i) minimize $f = x - y + z$

s.t $x + 2y + z \le 6$

$2x - y + z \le 8$, $x, y, z \ge 0$

(ii) minimize $f = x + 2y - z$

s.t $2x - y + 3z \le 12$

$x + 2y + 5z \le 20$

$-x + y - z \le 8$, $x, y, z \ge 0$

(iii) maximize $f = x + 4y + 2z + 6$

s.t $x + 2y + z \le 10$

$x - y + z \le 8$, $x, y, z \ge 0$

(iv) maximize $f = 2x + 4y + 4z - 3u$

s.t $x + y + z \le 4$

$x + 4y + u \le 8$, $x, y, z, u \ge 0$

(v) maximize $f = 5x + 2y + 3z + 8$

s.t $x + 5y + 2z \le 30$

$x - 5y - 6z \le 40$, $x, y, z \ge 0$

(4)Solve the following LP problems:

(i) maximize $f = 2x + y + 4z$

s.t $x + y + 2z \le 20$

$x + 3y + 2z \ge 6$

$2x + 3y + 2z = 18$, $x, y, z \ge 0$

(ii) maximize $f = x + 2y + z - u$

s.t $2x - y + z - 2u \leq 6$

$x + y - z + u \geq 4$, $x, y, z, u \geq 0$

(iii) minimize $f = -x + 2y$

s.t $x + y \leq 4$

$x - 2y \leq 6$, $x \geq 0$ and y unrestricted

(iv) minimize $f = 2x + 4y$

s.t $x + 5y \leq 80$

$4x + 2y \geq 20$

$x + y = 10$, $x, y \geq 0$

3- Vector Spaces

(1) If R is the set of real numbers and $R^2 = \{(x, y): x, y \in R\}$. Show that R^2 is vector space on R and write its base.

(2)If $V = \{(x, y): x, y \in R\}$, where $a(x, y) = (ay, ax), a \in R$. Show that V is not vector space.

(3) If M is the set of all square matrices of order 2x2. Show that M is vector space on R and write its base.

(4) If M is the set of all polynomials of degree 3. Show that M is vector space on R and write its base.

(5)If A is matrix of order 2x2 and $M = \{AX: X \in R^2\}$. Show that M is subspace of the vector space R^2.

(6)If A is matrix of order 3x2 and $M = \{AX: X \in R^2\}$. Show that M is subspace of the vector space R^3.

(7)If A is matrix of order 2x2 and $M = \{X \in R^2: AX = 0\}$. Show that M is subspace of the vector space R^2.

4- Linear Transformations

(1)Write the transformation of each matrix as $L : R^m \to R^n$:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix},\ B = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \end{bmatrix},\ C = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 5 \end{bmatrix},\ D = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix},\ E = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 2 & 1 & 3 & 4 \end{bmatrix},\ G = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

(2)Write the matrix of each transformation :

(i) L: $R^3 \to R^2$, defined by : $L(x_1, x_2, x_3) = (x_1 + x_2,\ x_1 - 2x_3)$

(ii) L: $R^2 \to R^3$, defined by : $L(x_1, x_2) = (x_1 + 2x_2,\ x_1, x_2 - x_1)$

(iii) L: $R^3 \to R^4$, defined by : $L(x_1, x_2, x_3) = (x_1 + x_2, x_1 - x_3, x_1 + x_2 - x_3, x_3 - x_2)$

(3)Determine the linear transformation among the following :

(i) L: $R^3 \to R^2$, defined by : $L(x_1, x_2, x_3) = (x_1 - x_2,\ x_1 + x_3)$

(ii)L: $R^3 \to R^2$, defined by : $L(x_1, x_2, x_3) = (x_1 + 1,\ x_2 + 2,\ x_1 + x_3)$

(iii) L: $R^2 \to R^3$, defined by : $L(x_1, x_2) = (x_1 + x_2,\ x_1,\ x_2 - x_1)$

(iv) L: $R^2 \to R^3$, defined by : $L(x_1, x_2) = (x_1 + x_2 - 2,\ x_1 + 3,\ x_2 - x_1)$

(4)Find the kernel of the following matrices :

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix},\ C = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix},\ D = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

5- Complex Analysis

(1)If $z_1 = 2 + 3i,\ z_2 = 1 - 3i,\ z_3 = 3 + 4i.$

Find $z_1 + z_2,\ z_2 - z_3,\ z_2.z_3,\ \sqrt[4]{z_1}.(z_2)^6,\ z_1 + z_2 + z_3,\ z_1.z_2.z_3$

(2)Put the following complex numbers in polar form :

(i) $z = 3 + 3i$ (ii) $z = 2i$ (iii) $z = -2 + 2i$

(iv) $z = -\frac{3}{4}i$ (v) $z = -\frac{2}{3}$ (vi) $z = \frac{1}{2} - \frac{1}{2}i$

(3)Put the following in rectangular form :

(i) $z = 3\,e^{\pi i}$ (ii) $z = 2^{\pi i}$ (iii) $z = e^{2-\pi i}$

(iv) $z = e^{\frac{1}{2}\pi i}$ (v) $z = e^{1+\frac{1}{4}\pi i}$ (vi) $z = e^{-1-\ln(2i)}$

(4)Solve the following equations :

(i) $\cos z = 3$ (ii) $\sin z = 2$ (iii) $\cosh z = 4$

(iv) $\ln(z^2 + 2) = \pi i$ (v) $(z^2 + 4)^2 = 0$ (vi) $e^{2z} = e^{1-z}$

(5)Determine and sketch the image of the following regions under the function $f(z) = \sin z$:

(i) $0 \le x \le 2\pi,\ 1 \le y \le 2$ (ii) $0 \le x \le \frac{\pi}{2},\ 0 \le y \le 2$

(iii) $-\frac{\pi}{2} \le x \le \frac{\pi}{2},\ 0 \le y \le 2$

(6)Sketch the image of the region : $0 \le x \le \pi,\ 0 \le y \le 1$ under the function $f(z) = \cos z$.

(7)Find the image of the following regions under the function $f(z) = e^z$

(i) $0 \le x \le 1,\ 0 \le y \le \frac{\pi}{2}$ (ii) $\ln 2 \le x \le \ln 3,\ 0 \le y \le \pi$

(8)Determine which of the following functions are harmonic. For each harmonic function find its conjugate such that f(z) is analytic:

(i) $u = x\sin y - y\cos x$ (ii) $v = 3 + x^2 - y^2$

(iii) $u = x^2 + 2y - y^2$ (iv) $v = x^2 + 2x - y^2$

(9)Find u(x, y) , v(x, y) of each of the following functions and show that they satisfy Remman's equations and Laplace equations :

(i) $f(z) = z + \sin 2z$ (ii) $f(z) = z^2 + 2\cosh 2z$

(iii) $f(z) = \ln 3 + \cos^2 z$ (iv) $f(z) = z + e^{2z}$

(10)Find the zeroes and their order of each of the following functions :

(i) $f(z) = z^4 + z^2$ (ii) $f(z) = z^4 - 16$ (iii) $f(z) = \frac{1}{z}\sin z^3$

(iv) $f(z) = e^{2z} - e^z$ (v) $f(z) = z(e^z - 1)$ (vi) $f(z) = z\cos z^2$

(11)Show that :

(i) $\operatorname*{Res}_{z=i} f(z) = \operatorname*{Res}_{z=-i} f(z) = \frac{1}{2}$ where $f(z) = \frac{z}{z^2+1}$

(ii) $\operatorname*{Res}_{z=\frac{\pi}{2}} f(z) = \operatorname*{Res}_{z=-\frac{\pi}{2}} f(z) = -1$ where $f(z) = \tan z$

(iii) $\operatorname*{Res}_{z=0} f(z) = \frac{1}{2}$ where $f(z) = \frac{1}{z + \sin z}$

(iv) $\operatorname*{Res}_{z=0} f(z) = 3$ where $f(z) = e^{\frac{3}{z}}$

(v) $\operatorname*{Res}_{z=0} f(z) = -\frac{1}{6}$ where $f(z) = z^2 \sin\frac{1}{z}$

(vi) $\operatorname*{Res}_{z=0} f(z) = 1$ where $f(z) = \frac{\sin z}{z^2}$

(vii) $\operatorname*{Res}_{z=0} f(z) = 0$ where $f(z) = \frac{\sin z}{z^3}$

(12)If C is the ellipse : $z = 5\cos t + i\,4\sin t$. Show that :

(i) $\oint_C \frac{1}{z+9}dz = 0$ (ii) $\oint_C \frac{e^{2z}}{z-3\pi i}dz = 0$ (iii) $\oint_C \frac{\cosh 2z}{z+9i}dz = 0$

(iv) $\oint_C \frac{\ln(z-7)}{z^2+36}dz = 0$ (v) $\oint_C \frac{\cos z}{z^2-49}dz = 0$ (vi) $\oint_C \frac{\sinh 2z}{z-9i}dz = 0$

(13)If C is the circle : $|z| = 1$. Show that :

(i) $\oint_C \frac{1}{z}dz = 2\pi i$ (ii) $\oint_C \frac{1}{4z+i}dz = \frac{\pi}{2}i$ (iii) $\oint_C \frac{\cos z}{z}dz = 2\pi i$

(iv) $\oint_C \frac{e^z}{z^2}dz = 2\pi i$ (v) $\oint_C \frac{z^2}{(2z-5)}dz = 0$ (vi) $\oint_C \frac{4^z}{2z-1}dz = 4\pi i$

(14)If C is the circle : $|z| = 4$. Show that :

(i) $\oint_C \frac{z}{z^2-1}dz = 2\pi i$ (ii) $\oint_C \frac{z+1}{z^2(z+2)}dz = 0$

(iii) $\oint_C \frac{z^2}{(z^2+3z+2)^2}dz = 0$ (iv) $\oint_C \frac{1}{z(z-2)^3}dz = 0$

(v) $\oint_C \frac{1}{z^2+z+1}dz = 0$ (vi) $\oint_C \frac{1}{(z+1)^3}dz = 0$

(vii) $\oint_C \frac{z+2}{z(z+1)}dz = 2\pi\, i$ (viii) $\oint_C \frac{1}{z(z+1)(z+4)}dz = -\frac{\pi}{6}\, i$

(15)If C is the circle : $|z| = 1$. Find the integrals :

(i) $\oint_C \frac{z^3}{(2z-1)^2}dz$ (ii) $\oint_C \frac{\sin z}{4z-\pi}dz$ (iii) $\oint_C \frac{\cos z}{(4z+\pi)^2}dz$

(iv) $\oint_C \frac{e^z}{z^3(2z+1)}dz$ (v) $\oint_C \frac{\ln(z+5)}{z^2}dz$ (vi) $\oint_C \frac{\cosh z}{z^4}dz$

(16) Show that :

(i) $\int_0^{2\pi} \frac{1}{10-6\sin\theta}d\theta = \frac{\pi}{4}$ (ii) $\int_0^{2\pi} \frac{1}{3+\cos\theta+2\sin\theta}d\theta = \pi$

(iii) $\int_0^{2\pi} \frac{\sin^2\theta}{5+4\cos\theta}d\theta = \frac{\pi}{4}$ (iv) $\int_0^{2\pi} \frac{1}{(5-3\sin\theta)^2}d\theta = \frac{5\pi}{32}$

(v) $\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}d\theta = \frac{4\pi}{\sqrt{27}}$ (vi) $\int_{-\infty}^{\infty} \frac{\cos x}{x(x^2-2x+2)}dx = \frac{\pi}{2}e^{-1+i}$

(vii) $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+9}dx = \frac{\pi}{e^3}$ (viii) $\int_{-\infty}^{\infty} \frac{1}{(x+1)^2(x^2+9)}dx = \frac{\pi}{12}$

(ix) $\int_{-\infty}^{\infty} \frac{x^2}{x^6+1}dx = \frac{\pi}{3}$ (x) $\int_{-\infty}^{\infty} \frac{\cos 2x}{(x^2+9)^2}dx = \frac{7\pi}{108\, e^6}$